3.2.11 Theorem *Let (x_n) be a sequence of positive real numbers such that $L := \lim(x_{n+1}/x_n)$ exists. If $L < 1$, then (x_n) converges and* $\lim(x_n) = 0$.

Proof. By 3.2.4 it follows that $L \geq 0$. Let r be a number such that $L < r < 1$, and let $\varepsilon := r - L > 0$. There exists a number $K \in \mathbb{N}$ such that if $n \geq K$ then

$$\left|\frac{x_{n+1}}{x_n} - L\right| < \varepsilon.$$

It follows from this (why?) that if $n \geq K$, then

$$\frac{x_{n+1}}{x_n} < L + \varepsilon = L + (r - L) = r.$$

Therefore, if $n \geq K$, we obtain

$$0 < x_{n+1} < x_n r < x_{n-1} r^2 < \cdots < x_K r^{n-K+1}.$$

If we set $C := x_K / r^K$, we see that $0 < x_{n+1} < C r^{n+1}$ for all $n \geq K$. Since $0 < r < 1$, it follows from 3.1.11(b) that $\lim(r^n) = 0$ and therefore from Theorem 3.1.10 that $\lim(x_n) = 0$. Q.E.D.

As an illustration of the utility of the preceding theorem, consider the sequence (x_n) given by $x_n := n/2^n$. We have

$$\frac{x_{n+1}}{x_n} = \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} = \frac{1}{2}\left(1 + \frac{1}{n}\right),$$

so that $\lim(x_{n+1}/x_n) = \frac{1}{2}$. Since $\frac{1}{2} < 1$, it follows from Theorem 3.2.11 that $\lim(n/2^n) = 0$.

Exercises for Section 3.2

1. For x_n given by the following formulas, establish either the convergence or the divergence of the sequence $X = (x_n)$.
   (a) $x_n := \dfrac{n}{n+1}$, (b) $x_n := \dfrac{(-1)^n n}{n+1}$,
   (c) $x_n := \dfrac{n^2}{n+1}$, (d) $x_n := \dfrac{2n^2+3}{n^2+1}$.
2. Give an example of two divergent sequences X and Y such that:
   (a) their sum $X + Y$ converges, (b) their product XY converges.
3. Show that if X and Y are sequences such that X and $X + Y$ are convergent, then Y is convergent.
4. Show that if X and Y are sequences such that X converges to $x \neq 0$ and XY converges, then Y converges.
5. Show that the following sequences are not convergent.
   (a) (2^n), (b) $\left((-1)^n n^2\right)$.
6. Find the limits of the following sequences:
   (a) $\lim\left((2 + 1/n)^2\right)$, (b) $\lim\left(\dfrac{(-1)^n}{n+2}\right)$,
   (c) $\lim\left(\dfrac{\sqrt{n}-1}{\sqrt{n}+1}\right)$, (d) $\lim\left(\dfrac{n+1}{n\sqrt{n}}\right)$.

7. If (b_n) is a bounded sequence and $\lim(a_n) = 0$, show that $\lim(a_n b_n) = 0$. Explain why Theorem 3.2.3 *cannot* be used.
8. Explain why the result in equation (3) before Theorem 3.2.4 *cannot* be used to evaluate the limit of the sequence $((1 + 1/n)^n)$.
9. Let $y_n := \sqrt{n+1} - \sqrt{n}$ for $n \in \mathbb{N}$. Show that $(\sqrt{n}y_n)$ converges. Find the limit.
10. Determine the limits of the following sequences.
    (a) $(\sqrt{4n^2+n} - 2n)$, (b) $(\sqrt{n^2+5n} - n)$.
11. Determine the following limits.
    (a) $\lim\left((3\sqrt{n})^{1/2n}\right)$, (b) $\lim\left((n+1)^{1/\ln(n+1)}\right)$.
12. If $0 < a < b$, determine $\lim\left(\dfrac{a^{n+1} + b^{n+1}}{a^n + b^n}\right)$.
13. If $a > 0, b > 0$, show that $\lim\left(\sqrt{(n+a)(n+b)} - n\right) = (a+b)/2$.
14. Use the Squeeze Theorem 3.2.7 to determine the limits of the following,
    (a) (n^{1/n^2}), (b) $\left((n!)^{1/n^2}\right)$.
15. Show that if $z_n := (a^n + b^n)^{1/n}$ where $0 < a < b$, then $\lim(z_n) = b$.
16. Apply Theorem 3.2.11 to the following sequences, where a, b satisfy $0 < a < 1, b > 1$.
    (a) (a^n), (b) $(b^n/2^n)$,
    (c) (n/b^n), (d) $(2^{3n}/3^{2n})$.
17. (a) Give an example of a convergent sequence (x_n) of positive numbers with $\lim(x_{n+1}/x_n) = 1$.
    (b) Give an example of a divergent sequence with this property. (Thus, this property cannot be used as a test for convergence.)
18. Let $X = (x_n)$ be a sequence of positive real numbers such that $\lim(x_{n+1}/x_n) = L > 1$. Show that X is not a bounded sequence and hence is not convergent.
19. Discuss the convergence of the following sequences, where a, b satisfy $0 < a < 1, b > 1$.
    (a) $(n^2 a^n)$, (b) (b^n/n^2),
    (c) $(b^n/n!)$, (d) $(n!/n^n)$.
20. Let (x_n) be a sequence of positive real numbers such that $\lim(x_n^{1/n}) = L < 1$. Show that there exists a number r with $0 < r < 1$ such that $0 < x_n < r^n$ for all sufficiently large $n \in \mathbb{N}$. Use this to show that $\lim(x_n) = 0$.
21. (a) Give an example of a convergent sequence (x_n) of positive numbers with $\lim(x_n^{1/n}) = 1$.
    (b) Give an example of a divergent sequence (x_n) of positive numbers with $\lim(x_n^{1/n}) = 1$. (Thus, this property cannot be used as a test for convergence.)
22. Suppose that (x_n) is a convergent sequence and (y_n) is such that for any $\varepsilon > 0$ there exists M such that $|x_n - y_n| < \varepsilon$ for all $n \geq M$. Does it follow that (y_n) is convergent?
23. Show that if (x_n) and (y_n) are convergent sequences, then the sequences (u_n) and (v_n) defined by $u_n := \max\{x_n, y_n\}$ and $v_n := \min\{x_n, y_n\}$ are also convergent. (See Exercise 2.2.18.)
24. Show that if (x_n), (y_n), (z_n) are convergent sequences, then the sequence (w_n) defined by $w_n := \text{mid}\{x_n, y_n, z_n\}$ is also convergent. (See Exercise 2.2.19.)

Section 3.3 Monotone Sequences

Until now, we have obtained several methods of showing that a sequence $X = (x_n)$ of real numbers is convergent: